A FACTORISATION AND AN EQUIVALENCE FOR OPERATORS OF CONVOLUTION TYPE

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1 A problem of the factorization

Let

$$a_n u^{(n)}(z) + a_{n-1} u^{(n-1)}(z) + \ldots + a_0 u(z) = 0 \qquad (1)$$

be an ordinary differential equation with constant coefficients,

$$P(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0$$

the characteristic polynomial of this equation. Let us assume that $P(\lambda)$ admits a representation

$$P(\lambda) = P_1(\lambda) P_2(\lambda),$$

where $P_1(\lambda)$, $P_2(\lambda)$ are polynomials having no common zeros. Evidently, every solution $u(z)$ of the equation (1) can be written as

$$u(z) = u_1(z) + u_2(z),$$

where $u_1(z)$ is a solution of a differential equation with the characteristic polynomial $P_1(\lambda)$ and $u_2(z)$ is a solution of a homogeneous differential equation with the characteristic polynomial $P_2(\lambda)$. The problem of a factorisation was set up by J. Adamar for the case of partial differential equations of finite order with constant coefficients. In this case, an additive factorisation of solutions of homogeneous equations is always possible [1], [2].

Napalkov has demonstrated, (see, for example, [3], p. 210), that for convolution type operators such a factorisation is not always possible. He solved the following problem.

Let us consider an equation

$$M_L(u) = \sum_{k=0}^{\infty} c_k u^{(k)}(z) = 0, \qquad (2)$$

whose characteristic function $L(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k$ is an entire function of exponential type. Let us assume that $L(\lambda) = L_1(\lambda) L_2(\lambda)$, where $L_1(\lambda)$, $L_2(\lambda)$ are also entire functions of exponential type. In what case any entire solution $u(z)$ of equation (2) can be written in the form

$$u(z) = u_1(z) + u_2(z), \quad M_{L_j}(u_j) = 0 \quad (j = 1,\ 2)? \qquad (3)$$

Theorem 1. *Any entire solution $u(z)$ of equation (2) can be represented in the form (3) if and only if*

$$1 = L_1(\lambda)\varphi_1(\lambda) + L_2(\lambda)\varphi_2(\lambda), \tag{4}$$

where $\varphi_1(\lambda)$, $\varphi_2(\lambda)$ are some entire functions of exponential type.

Let $\gamma(t) = \sum\limits_{k=0}^{\infty} c_k k!/t^{k+1}$ be the function associated according to Borel with the function $L(\lambda)$ and let $\overline{D}$ be the adjoint diagram of the function $L(\lambda)$. Then, by the Cauchy formula, the equation (2) can be represented as

$$M_L(u) = \frac{1}{2\pi i} \int\limits_C \gamma(t)u(t+z)dt = 0, \tag{5}$$

where C is a closed contour surrounding $\overline{D}$. The equation (5) is a convolution equation.

A deduction of condition (4) uses algebraic and functional methods.

In the paper [4] Korobeynik obtained some conditions for factorisation of convolution operators, using non-trivial decompositions of zero with respect to a system of exponents. Moreover, he extended the "sufficient part" of Theorem 1 to a rather general situation of generalized convolution operators.

The proof of sufficiency of condition (4) may be obtained, for example, from the relation

$$u(z) = M_1(u) = M_{L_1\varphi_1}(u) + M_{L_2\varphi_2}(u) = u_2(z) + u_1(z).$$

Using the theorem on approximation of an entire solution $u(z)$ by means of elementary solutions and also the continuity of the operators

$$M_{L_1\varphi_1},\ M_{L_2\varphi_2},\ M_{L_1},\ M_{L_2},$$

one can show that

$$M_{L_j}(u_j) = 0.$$

Further, Theorem 1 can be easily transferred to the case when the characteristic function has an arbitrary number of factors:

$$L(\lambda) = L_1(\lambda) \cdot \ldots \cdot L_m(\lambda), \quad m \geqslant 2.$$

Then the condition (4) can be rewritten as

$$1 = \sum_{j=1}^{m} N_j(\lambda) \cdot \varphi_j(\lambda), \quad \text{where} \quad N_j(\lambda) = \frac{L(\lambda)}{L_j(\lambda)}.$$

Now let the characteristic function $L(\lambda)$ of equation (2) be an entire function of class $[1,\ 0]$, i.e. it has order $\leqslant 1$ and it is of minimal type. Let us assume that $L(\lambda)$ has an infinite number of zeros and all they are simple. It is known that in this case the operator M_L can be applied to any analytical function at points of its regularity, any analytic solution of (2) is univalent everywhere and its domain D is convex. Let us assume that $L(\lambda) = L_1(\lambda)L_2(\lambda)$, where $L_1(\lambda)$ and $L_2(\lambda)$ belong to $[1,\ 0]$. The following theorems hold ([5]).

Theorem 2. *Let us assume that there exist functions φ_1, φ_2 in $[1,0]$ with property (4). Then any analytic solution $u(z)$ of equation (2) can be represented in the form (3), and if $u(z)$ is analytic on D, then the functions $u_j(z)$ are analytic on D.*

Theorem 3. *Let us assume that for any disk K with the center at zero, any analytic solution of equation (2) on K has the form (3), where $u_j(z)$ are analytic functions on K. Then there exist functions φ_1, φ_2 in $[1,0]$ with property (4).*

Thus, if for any disk K with the center at zero, any solution $u(z)$ of equation (2) analytic on K has the form (3), then representation (3) is true for any analytic solution on D .

In connection with this the following question arose. Let us fix a disk K with the center at zero and assume that any solution $u \in H(K)$ has the form (3). Would any solution $u(z) \in H(K_1)$ $K_1 \subset K$ have the form (3)? In [6] a counterexample of an entire function of exponential type $L(\lambda) = L_1(\lambda)L_2(\lambda)$ is constructed.

2 Case of generalized derivatives

Let

$$f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$$

be an entire function of order ρ and type $\sigma \neq 0,\ \infty$ with $\alpha_k \neq 0$. Let us assume that there exists the limit

$$\lim_{k\to\infty} k^{1/\rho}|\alpha_k|^{1/k} = (\sigma e \rho)^{1/\rho}. \tag{6}$$

Further, let

$$u(z) = \sum_{k=0}^{\infty} u_k z^k$$

be an arbitrary function analytic on the disk $U = \{z : |z| < R\}$, $0 < R \leqslant \infty$. The function

$$(D_f^n u)(z) = \sum_{k=0}^{\infty} u_{k+n} \frac{\alpha_k}{\alpha_{k+n}} z^k \tag{7}$$

is called the generalized derivative in the sense of Gelfond-Leontyev of order n of the function $u(z)$ generated by the function $f(z)$ [7].

According to condition (6), there exists the limit

$$\lim_{k\to\infty} \left|\frac{\alpha_k}{\alpha_{k+n}}\right|^{1/k} = 1,$$

hence the series (7) converges on the disk U and the function $D_f^n u$ is analytic on the same disk.

Notice the following easily checked properties of the operator (7):

1) $D_f^n(u_1 + u_2) = D_f^n u_1 + D_f^n u_2$;

2) if C is a constant, then $D_f^n(Cu) = CD_f^n(u)$;

3) $D_f^m(D_f^n u) = D_f^{m+n} u$;

4) $D_f^n(f(\lambda z)) = \lambda^n f(\lambda z)$, where λ is a constant;

5) if $f(z) = e^z$, then $D_f^n = d^n/dz^n$;

these properties show that the operator D_f^n is really a kind of derivative of order n;

6) the operator D_f^n is continuous: if a sequence $u_m(z)$ converges to $u(z)$ uniformly inside of U, then the sequence $D_f^n(u_m)$ converges to $D_f^n u$ uniformly inside of U, see [7], p. 61.

Let us consider a special case of the generating function:

$$f(z) = \sum_{k=0}^{\infty} \alpha_k z^k = 1 + \sum_{k=1}^{\infty} \frac{z^k}{p(1) \cdot \ldots \cdot p(k)}, \tag{8}$$

where $p(x) = a_p x^p + \cdots + a_1 x$ is a polynomial such that $p(k) \neq 0\,(k \in \mathbb{N})$. For large k we have

$$\alpha_k \sim \frac{1}{a_p^k \cdot (k!)^p},$$

whence property (6) holds with $\rho = 1/p$, $\sigma = p|a_p|^{1/p}$. The function $f(z)$ is an entire function of order $1/\rho$ and type σ. For $p(x) = x$ we have $f(z) = e^z$. If $f(z)$ is given by (8), then

$$D_f^n u(z) = \sum_{k=n}^{np} \frac{\Delta_k^{(n)}}{k!} z^{k-n} u^{(k)}(z), \tag{9}$$

where ([7], p. 75)

$$\begin{aligned} \Delta_k^{(n)} &= \varphi_n(k) - C_k^1 \varphi_n(k-1) + \ldots + (-1)^k \varphi_n(0), \\ \varphi_n(x) &= p(x)p(x-1)\ldots p(x-n+1). \end{aligned}$$

Notice that generalized derivatives (7) are defined only for functions analytic on a neighbourhood of zero. If $f(z)$ is given by (8), then generalized derivatives (9) are defined at all points where $u(z)$ is analytic.

Let us consider now the equation

$$M_L(u) = \sum_{k=0}^{\infty} c_k D_f^n u = 0, \tag{10}$$

whose characteristic function $L(\lambda)$ has order ρ_1, let us assume that $\rho_1 > \rho$ and that all zeros of $L(\lambda)$ are simple. It is known, that the series in (10) converges uniformly on any bounded domain, when the order ν of $u(z)$ satisfies the condition

$$\nu < \frac{\rho \rho_1}{\rho_1 - \rho}. \tag{11}$$

Let us assume that $L(\lambda) = L_1(\lambda) L_2(\lambda)$, where $L_j(\lambda)$ belongs to $[\rho_1, \infty]$. Leontiev ([8]) proved:

Theorem 4. *Any solution $u(z)$ (with order ν satisfying condition (11)) can be represented in the form (3) if and only if condition (4) with $\varphi_j(\lambda) \in [\rho_1, \infty]$ is satisfied.*

Now let us fix a number h such that

$$\rho < h < \frac{\rho \rho_1}{\rho_1 - \rho}.$$

In the paper [10] the problem of factorization has been solved for the following subclasses of entier functions $u(z)$:
1) having order $< h$,
2) $[h, \infty]$,
3) $[h, \infty)$.
Criterions have the form (4).

A very general definition for the operator of a generalized differentiation was given by Leontyev in 1965 ([9]). Let $\{P_k(z)\}$ be some basis system of functions in $H(D)$. Then, if $u(z)$ is represented uniquely as a series $u(z) = \sum_{k=0}^{\infty} d_k P_k(z)$, convergent on D (or on its part), then one defines $D_1^n u(z) = \sum_{k=n}^{\infty} d_k P_{k-n}(z)$. In particular, for $P_n(z) = a_n z^n$, one has the Gelfond-Leontyev operator D_f^n.

In [11] under some additional conditions on $\{P_n(z)\}$, the problem of factorisation has been solved for an operator of infinite order in terms of derivatives D_1^n. For example, these conditions are satisfied by Faber's polynomials and functions in Fage's basis, constructed by means of the differential operator with entier coefficients.

3 Equivalence of differential operator

In various places of the theory of functions, functional analysis and theory of differential equations one often runs into problems of an equivalence of operators.

Let A_R $(0 < R \leqslant \infty)$ be the space of all univalent and analytic functions on the disk $\{z \in \mathbb{C} : |z| < R\}$ and $[\rho, \sigma)$ the space of all entier functions of order $< \rho$, or of order $\leqslant \rho$ and of type $< \sigma$.

Let E_1 and E_2 be locally convex spaces over the field $\mathbb{C}$. Linear continuous operators A on E_1 and B on E_2 are called *equivalent*, if there exists a linear one-to-one and mutually continuous map $T : E_1 \to E_2$ such that $TA = BT$.

Let us consider a differential equation of infinite order with constant coefficients

$$M_L(u) := \sum_{k=0}^{\infty} a_k D^k u = 0,$$

whose characteristic function $L(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k$ is an entire function of exponential type σ.

In [12] there was proved that $M_L(u)$ is equivalent to an operator of finite order $D^p(u) = \sum_{k=0}^{p} d_k D^k u$. If $L(\lambda) \in [1, \infty)$, then for equivalence of these operators in the space A_∞, it is necessary and sufficient that $M_L(u) = \sum_{k=0}^{p} a_k D^k u$ with $a_p \neq 0$.

In this work [12] the following problem was set up. One has to find necessary and sufficient conditions for equivalence of continuous differential operators of infinite order with constant coefficients

$$M_{L_1}(u) = \sum_{k=0}^{\infty} a_k D^k u \quad \text{and} \quad M_{L_2}(u) = \sum_{k=0}^{\infty} b_k D^k u,$$

where

$$L_1(\lambda) = \sum_{k=0}^{\infty} a_k \lambda^k \quad \text{and} \quad L_2(\lambda) = \sum_{k=0}^{\infty} b_k \lambda^k$$

are entire functions of class $[1, \infty)$.

In some cases one succeeded in establishing an equivalence of such operators. For example, in the paper [12] it was done for operators e^{aD} and e^{bD}. They are equivalent if and only if either $a = b$ or $ab \neq 0$.

This result was transferred by the authors to operators of infinite order with various generalized derivatives. Moreover, as a co-product, they obtained similar results for some operators of finite order.

Let entire functions $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ and $g(z) = \sum_{k=0}^{\infty} \beta_k z^k$ have order $\rho \in (0, \infty)$ and types $\sigma \in (0, \infty)$ and $\sigma_1 \in (0, \infty)$ respectively, and $\alpha_k \neq 0$, $\beta_k \neq 0$, $k = 0, 1, \ldots$, and the following conditions be satisfied:

$$\lim_{k\to\infty} k^{1/\rho} \sqrt[k]{|\alpha_k|} = (\sigma e\rho)^{1/\rho}, \quad \lim_{k\to\infty} k^{1/\rho} \sqrt[k]{|\beta_k|} = (\sigma_1 e\rho)^{1/\rho}. \tag{12}$$

Consider generalized derivatives in the sense of Gelfond-Leontyev of a function $u(z) = \sum_{k=0}^{\infty} u_k z^k$ in A_R generated by functions $f(z)$ and $g(z)$.

It is known, ([7], p. 62) that these derivatives are equivalent in A_R. An isomorphism T of the space A_R is the following:

$$T\Big(\sum_{k=0}^{\infty} u_k z^k\Big) = \sum_{k=0}^{\infty} u_k \frac{\beta_k}{\alpha_k} z^k.$$

The paper [13] contains the following results.

Theorem 5. *If $a,\ b \in \mathbb{C}$ and $ab \neq 0$, then $e^{aD_f} \sim e^{bD_f}$ in A_∞.*

Here

$$Tu(z) = u\Big(\frac{a}{b} z\Big).$$

Theorem 6. *Let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ be an entire function such that $\alpha_k \neq 0$, $k = 0, 1, 2, \ldots$, and*

$$\lim_{k\to\infty} \sqrt[k]{|\alpha_k k!|} = \sigma = \frac{|b|}{|a|} \quad (\sigma \neq 0,\ \infty).$$

If $ab \neq 0$, then $e^{aD} \sim e^{bD_f}$ in A_R $(R \leqslant \infty)$.

Here

$$Tu(z) = \sum_{k=0}^{\infty} u_k \alpha_k k! \Big(\frac{a}{b}\Big)^k z^k.$$

Theorem 7. *Let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ and $g(z) = \sum_{k=0}^{\infty} \beta_k z^k$ be entire functions of order $\rho \in (0, \infty)$ and types $\sigma \neq 0,\ \infty$ and $\sigma_1 \neq 0,\ \infty$ respectively, with $\alpha_k \neq 0$, $\beta_k \neq 0$, $k = 0, 1, 2, \ldots$, conditions (12) be satisfied, and*

$$\lim_{k\to\infty} \sqrt[k]{\left|\frac{\beta_k}{\alpha_k}\right|} = \frac{|b|}{|a|}.$$

If $ab \neq 0$, then $e^{aD_f} \sim e^{bD_g}$ in A_R $(R \leqslant \infty)$.

Here

$$Tu(z) = \sum_{k=0}^{\infty} k \frac{\beta_k}{\alpha_k} \left(\frac{a}{b}\right)^k z^k.$$

In the paper [14] the authors considered, instead of the differential operator D, an operator of the generalized differentiation $\overline{D}$, introduced by Korobeynik, and the differential operator D_f.

Let $u(z) = \sum_{k=0}^{\infty} u_k z^k$ belong to A_R. Operators $\overline{D}^n$, $n \in \mathbb{N}$, are defined by

$$\overline{D}^n(u(z)) = \sum_{k=0}^{\infty} c_k \cdot \ldots \cdot c_{k+n-1} u_{k+n} z^k,$$

where $\{c_k\}$ is a sequence of complex numbers such that

$$\overline{\lim_{k\to\infty}} |c_k|^{1/k} \leqslant 1.$$

If $c_k = k+1$, then $\overline{D}^n = D^n$.

Theorem 8. *Let $f(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ be an entire function of order $\rho \in (0, \infty)$ and type $\sigma \neq 0, \infty$ with $\alpha_k \neq 0$, $k = 0, 1, \ldots$, and $\lim\limits_{k\to\infty} k^{1/\rho} |\alpha_k|^{1/k} = (\sigma e \rho)^{1/\rho}$. Let $a, b \in \mathbb{C}$. If $ab \neq 0$, then $e^{a\overline{D}} \sim e^{bD_f}$ in A_∞.*

Here

$$Tu(z) = T\Big(\sum_{k=0}^{\infty} u_k z^k\Big) = u_0\alpha_0 + \sum_{k=1}^{\infty} \frac{a^k}{b^k} u_k \alpha_k c_0 \cdot \ldots \cdot c_{k-1} z^k.$$

In the paper [15] the authors established the equivalence of generalized derivatives in the sense of Gelfond-Leontyev generated by functions of a special form.

Theorem 9. *Let $p(x)$ and $q(x)$ be polynomials:*

$$p(x) = \alpha_p x^p + \cdots + \alpha_1 x, \quad q(x) = \beta_p x^p + \cdots + \beta_1 x,$$

such that $p(k) \neq 0$ and $q(k) \neq 0$ for $k \in \mathbb{N}$. If

$$p(1) \cdot p(2) \cdot \ldots \cdot p(n) = q(1) \cdot q(2) \cdot \ldots \cdot q(n),$$

then

$$\sum_{k=n}^{np} \frac{\Delta_k^{(n)}}{k!} z^{k-n} u^{(k)}(z) \sim \sum_{k=n}^{np} \frac{\delta_k^{(n)}}{k!} z^{k-n} u^{(k)}(z)$$

in A_R, $0 < R \leqslant \infty$, where

$$\begin{aligned}
\Delta_k^{(n)} &= \varphi_n(k) - C_k^1 \varphi_n(k-1) + \cdots + (-1)^k \varphi_n(0),\\
\varphi_n(x) &= p(x)p(x-1)\ldots p(x-n+1),\\
\delta_k^{(n)} &= \psi_n(k) - C_k^1 \psi_n(k-1) + \cdots + (-1)^k \psi_n(0),\\
\psi_n(x) &= q(x)q(x-1)\ldots q(x-n+1).
\end{aligned}$$

Now let us consider operators

$$M_L u(z) = \sum_{n=0}^{\infty} d_n D_f^n u(z)$$

and

$$M_L^1 u(z) = \sum_{n=0}^{\infty} d_n D_g^n u(z),$$

where generating functions f and g have a special form and

$$L(\lambda) = \sum_{n=0}^{\infty} d_n \lambda^n \in [1,\ R).$$

Using the scheme of the paper [11] and Theorem 9, we proved the following theorem.

Theorem 10. *Let $p(x)$ and $q(x)$ be polynomials:*

$$p(x) = \alpha_p x^p + \cdots + \alpha_1 x, \quad q(x) = \beta_p x^p + \cdots + \beta_1 x,$$

such that $p(k) \neq 0$ and $q(k) \neq 0$ for $k \in \mathbb{N}$. If $p(1) = q(1)$, then
1. $M_L \sim M_L^1$ in A_R, $0 < R \leqslant \infty$;
2. if there exists a factorisation for the equation $M_L u(z) = 0$, then it is true for the equation $M_L^1 u(z) = 0$.

References

[1] V. P. Palamodov. Linear differential operators with constant coefficients. Moscow: Nauka, 1967.

[2] S. Matzuura. Factorisation of differential operators and decomposition of homogeneous equations. Osaka J. Math., 1963, vol. 15, No. 2, 213–231.

[3] V. V. Napalkov. Convolution equations in many-dimensional vector spaces. Moscow: Nauka, 1982.

[4] Ju. F. Korobeynik. On some applications of non-trivial decompositions of zero in the theory of convolutions. Dokl. Akad. Nauk USSR, 1990, tom 313, No. 6, 1304–1328.

[5] A. Yu. Timofeev. On the representation of a solution of an equation of infinite order as the sum of two solutions. Matem. zam., 1982, tom 31, No. 2, 245–256.

[6] A. Yu. Timofeev. A contribution to a problem of the factorization for a differential operator of infinite order. In: "The theory of functions and approximations". 3-rd Saratov Winter School, Saratov Univ., 1988, Part 3, 66–68.

[7] A. F. Leontyev. Generalizations of series of exponents. Moscow: Nauka, 1981.

[8] A. F. Leontyev. On an application of the interpolation method. Matem. zam., 1975, tom 18, No. 5, 735–752.

[9] A. F. Leontyev. On a functional equation. Izv. Akad.Nauk USSR, Ser. matem., 1965, tom 29, No. 4, 725–756.

[10] A. Yu. Timofeev. Decomposition of sequences of polynomials of exponents into subsequence. Kand. diss., Ufa, 1981.

[11] A. Yu. Timofeev. Factorisierung des Differentialoperators unendlicher Ordnung in verallgemeinerten Ableitungen. Zeitschr. Anal. Anwend., 1989, Bd. 8, No. 2, 177–181.

[12] M. K. Fage, N. I. Nagnibida. The problem of an equivalence of ordinary differential operators. Novosibirsk: Nauka, 1987.

[13] V. Ju. Briuts, A. Yu. Timofeev. About equivalence of differential operators in Gelfond-Leontyev's derivatives. Materials of the VIII Intern. Sci. Kravchouk Conference. Kyiv, 2000, 246.

[14] V. Ju. Briuts, A. Yu. Timofeev. On equivalence of differential operators with generalized derivatives. Complex analysis, differential equaqions and related topics. I. Complex analysis. In-t Mat. with Comp. Cent., Ural Mat. Cent. RAN, Ufa, 2000, 27–30.

[15] V. Ju. Briuts, A. Yu. Timofeev. On equivalence of differential operators of finite order. Math. Vestnik Pedvuz. Volgo-Vyat. Reg., 2002, tom 4, 3–5.